

ON THE MONODROMY AT ISOLATED SINGULARITIES OF WEIGHTED HOMOGENEOUS POLYNOMIALS

BY

BENJAMIN G. COOPER

ABSTRACT. Assume $f: \mathbb{C}^m \rightarrow \mathbb{C}$ is a weighted homogeneous polynomial with isolated singularity, and define $\phi: S^{2m-1} - f^{-1}(0) \rightarrow S^1$ by $\phi(\bar{z}) = f(\bar{z})/|f(\bar{z})|$. If the monomials of f are algebraically independent, then the closure \bar{F}_0 of $\phi^{-1}(1)$ in S^{2m-1} admits a deformation into the subset G where each monomial of f has nonnegative real values. For the polynomial $f(z_1, \dots, z_m) = z_1^{a_1}z_2 + \dots + z_m^{a_m-1}z_m + z_m^{a_m}z_1$, G is a cell complex of dimension $m-1$, invariant under a characteristic map h of the fibration ϕ , and the inclusion $G \rightarrow \bar{F}_0$ induces isomorphisms in homology. To compute the homology of the link $K = f^{-1}(0) \cap S^{2m-1}$ it thus suffices to calculate the action of h_* on $H_{m-1}(G)$. Let $d = a_1a_2 \cdots a_m + (-1)^{m-1}$. Let w_1, w_2, \dots, w_m be the weights associated with f , satisfying $a_j/w_j + 1/w_{j+1} = 1$ for $j = 1, 2, \dots, m-1$ and $a_m/w_m + 1/w_1 = 1$. Let $n = d/w_1$, $q = \gcd(n, d)$, $r = q + (-1)^m$. Then $H_{m-2}(K) = \mathbb{Z}^r \oplus \mathbb{Z}_{d/q}$ and $H_{m-1}(K) = \mathbb{Z}^r$.

The purpose of this paper is to calculate for the complex polynomial f defined by

$$f(z_1, z_2, \dots, z_m) = z_1^{a_1}z_2 + z_2^{a_2}z_3 + \dots + z_m^{a_m-1}z_m + z_m^{a_m}z_1$$

the integral homology of the $(2m-3)$ -manifold K defined by $K = \{(z_1, \dots, z_m) \in \mathbb{C}^m: |z_1|^2 + \dots + |z_m|^2 = 1 \text{ and } f(z_1, \dots, z_m) = 0\}$. This contributes to a project begun by Milnor [3] and continued by Milnor and Orlik [4], Orlik and Wagreich [9], Orlik [6], Orlik and Randell [8], and others, to compute invariants which will help to describe the topology of a hypersurface defined by a complex polynomial near an isolated singularity. The results of this paper can be described briefly. Let $d = a_1a_2 \cdots a_m + (-1)^{m-1}$, and let q be the greatest common divisor of d and $a_2a_3 \cdots a_m - a_3a_4 \cdots a_m + a_4 \cdots a_m - \dots + (-1)^{m-2}a_m + (-1)^{m-1}$. Then K is $(m-3)$ -connected, $H_{m-2}(K)$ is the direct sum of a free abelian group of rank $q + (-1)^m$ and a cyclic group of order d/q , and $H_{m-1}(K)$ is free abelian of rank $q + (-1)^m$.

I am grateful to Peter Orlik for the survey article [7] which brought the problem to my attention and which has a good bibliography on this and related subjects.

1. Preliminaries. Let $f: \mathbb{C}^m \rightarrow \mathbb{C}$ be a polynomial, and let $V = f^{-1}(0)$. If V has a singularity at \bar{z}_0 , that is, all the partial derivatives of f vanish there, let S_ϵ denote the $(2m-1)$ -sphere in \mathbb{C}^m of radius ϵ , centered at \bar{z}_0 , and write $K = V \cap S_\epsilon$. Then for all ϵ sufficiently small, the portion of V inside S_ϵ is homeomorphic to the cone over K [3, p. 18], and so we are interested in the topology of K .

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Milnor showed in [3] that the function $\phi: S_\epsilon - K \rightarrow S^1$ defined by $\phi(\bar{z}) = f(\bar{z})/|f(\bar{z})|$ is the projection of a locally trivial fibre space and that if the singularity \bar{z}_0 is isolated (and ϵ is sufficiently small), then each fibre $F_\theta = \phi^{-1}(e^{i\theta})$ has the homotopy type of a bouquet of $(m-1)$ -spheres. The number of spheres involved, called μ , is the multiplicity of \bar{z}_0 as a zero of the gradient of f , a measure of the degeneracy of the critical point. Furthermore, still assuming the singularity is isolated, the closure $\overline{F_\theta}$ is a smooth $(2m-2)$ -dimensional manifold with boundary whose interior is F_θ and whose boundary is K . The manifold K is $(m-3)$ -connected.

Because $S_\epsilon - K$ is fibred over a circle, there exist one-parameter families of homeomorphisms $h_t: F_0 \rightarrow F_t$ for $0 \leq t \leq 2\pi$, with h_0 the identity map. Choose such a family and let h denote $h_{2\pi}$. The map h is called a characteristic map for the fibration, and the induced automorphism $h_*: H_{m-1}F_0 \rightarrow H_{m-1}F_0$, which is independent of the choice of family $\{h_t\}$, is called the monodromy of f at the singularity \bar{z}_0 . When $m \geq 3$, by using Alexander and Poincaré duality to substitute in the Wang sequence of the fibration, we obtain an exact sequence

$$0 \rightarrow H_{m-1}K \rightarrow H_{m-1}F_0 \xrightarrow{I_* - h_*} H_{m-1}F_0 \rightarrow H_{m-2}K \rightarrow 0,$$

where I is the identity map of F_0 [3, pp. 67–68]. Hence in the isolated singularity case, h_* determines the homology of K .

Let w_1, w_2, \dots, w_m be positive rational numbers. The polynomial $f(z_1, z_2, \dots, z_m)$ is *weighted homogeneous* with weights (w_1, w_2, \dots, w_m) if it can be expressed as a linear combination of monomials $z_1^{e_1} z_2^{e_2} \cdots z_m^{e_m}$ for which $e_1/w_1 + e_2/w_2 + \cdots + e_m/w_m = 1$. For a weighted homogeneous polynomial, the only possible isolated critical point is the origin. Weighted homogeneous polynomials for which the origin is an isolated critical point play an important role in Arnold's theory of normal forms for singularities (see, for example, [1]). Assume from now on that f is such a polynomial.

Then the radius ϵ of the sphere S_ϵ is immaterial. Specifically, from the evident identity

$$f(z_1 \cdot \exp\{w/w_1\}, z_2 \cdot \exp\{w/w_2\}, \dots, z_m \cdot \exp\{w/w_m\}) = e^w f(z_1, z_2, \dots, z_m)$$

we obtain both an explicit formula $h(z_1, z_2, \dots, z_m) = (z_1 \cdot \exp\{2\pi i/w_1\}, z_2 \cdot \exp\{2\pi i/w_2\}, \dots, z_m \cdot \exp\{2\pi i/w_m\})$ for a characteristic map and, for any positive ϵ and ϵ' , a diffeomorphism $S_\epsilon \rightarrow S_{\epsilon'}$ which is equivariant with respect to h , which maps $K = V \cap S_\epsilon$ to $K' = V \cap S_{\epsilon'}$, and which restricts to a fibre-preserving map $S_\epsilon - K \rightarrow S_{\epsilon'} - K'$. (Compare [9, p. 218].) We shall in what follows take $\epsilon = 1$ and take for characteristic map the map h defined just above.

Milnor and Orlik [4, p. 387] showed that the multiplicity μ , which equals the rank of $H_{m-1}F_0$, is $(w_1 - 1)(w_2 - 1) \cdots (w_m - 1)$.

Following [8], let us call a weighted homogeneous polynomial $g(z_1, \dots, z_k)$ with isolated critical point *decomposable* if, after suitable renumbering of the variables, $g(z_1, \dots, z_k) = g_1(z_1, \dots, z_l) + g_2(z_{l+1}, \dots, z_k)$, where g_1 and g_2 have isolated critical points at the origins of \mathbb{C}^l and \mathbb{C}^{k-l} , respectively. Oka showed in [5] that the Milnor fibre for g is then homotopy equivalent to the reduced join of the fibres for

g_1 and g_2 . The monodromy of g is therefore the tensor product of the monodromies of g_1 and g_2 . The polynomial which is the subject of this paper is indecomposable in the sense above, except in the degenerate case $a_1 = a_2 = \cdots = a_m = 1$ (see §2).

The main purpose of this paper is thus to compute the kernel and cokernel of $I_* - h_*: H_{m-1}F_0 \rightarrow H_{m-1}F_0$ for the weighted homogeneous polynomial $f(z_1, \dots, z_m) = z_1^{a_1}z_2 + \cdots + z_{m-1}^{a_{m-1}}z_m + z_m^{a_m}z_1$, assuming $a_1a_2 \cdots a_m > 1$. The computation is in two parts. First, the closure $\overline{F_0}$ is deformed into a subspace G , invariant under the explicit characteristic map h described above. The subspace G is the set where *each monomial* of f separately has the same argument, i.e., is real and nonnegative. The deformation exists for a whole class of weighted homogeneous polynomials, and its construction occupies §§4–14. In the case at hand, G is a cell complex of dimension $m - 1$. To calculate $h_*: H_{m-1}G \rightarrow H_{m-1}G$, and hence the homology of K , it is useful to filter G by the number of nonzero complex coordinates and study the resulting spectral sequence. The filtration corresponds to a sort of cellular decomposition of G which is described in §15. The remaining §§16–22 contain the actual calculations.

2. First let us dispose of the degenerate case. Suppose $g(z_1, \dots, z_m) = z_1z_2 + z_2z_3 + \cdots + z_{m-1}z_m + z_mz_1$. If $m \equiv 0 \pmod{4}$, then the critical point at the origin is not isolated. In all other cases we can make linear changes of variable to decompose g in the sense above. If m is odd, we may, for example, take $z'_1 = z_1 + z_3$, $z'_3 = z_3 + z_5$, \dots , $z'_{m-2} = z_{m-2} + z_m$, $z'_m = z_m$, $z'_2 = z_2 + z_m$, $z'_4 = z_4 - z_m$, \dots , $z'_{m-1} = z_{m-1} + (-1)^{(m-3)/2}z_m$. Then $g(z_1, z_2, \dots, z_m) = z'_1z'_2 + z'_3z'_4 + \cdots + z'_{m-2}z'_{m-1} + (-1)^{(m-1)/2}(z'_m)^2$. If $m \equiv 2 \pmod{4}$, we may take $z'_1 = z_1 + z_3$, $z'_3 = z_3 + z_5$, \dots , $z'_{m-1} = z_{m-1} + z_1$, and $z'_{2k} = z_{2k}$ for all k . Then $g(z_1, z_2, \dots, z_m) = z'_1z'_2 + z'_3z'_4 + \cdots + z'_{m-1}z'_m$. Therefore let us assume from now on that $a_1a_2 \cdots a_m > 1$.

3. In addition to the continuing assumption that $f(z_1, z_2, \dots, z_m)$ is a weighted homogeneous polynomial with isolated critical point, consider from now on the following additional assumption. Write f as a sum $f_1 + f_2 + \cdots + f_r$ of monomials, say

$$f_j(z_1, z_2, \dots, z_m) = c_j z_1^{e_{j,1}} z_2^{e_{j,2}} \cdots z_m^{e_{j,m}},$$

where c_j is a complex constant, and assume that the m -tuples $(e_{j,1}, e_{j,2}, \dots, e_{j,m})$ of integers for $j = 1, 2, \dots, r$, considered as vectors in Q^m , form an independent set. This condition has considerable overlap with the continuing assumption; it implies the existence of weights (w_1, w_2, \dots, w_m) with the desired homogeneity property, though not their positivity (nor their uniqueness if the number r of monomials is less than m). It rules out polynomials such as $f(z_1, z_2) = z_1^2 + z_1z_2 + z_2^2$, but it does apply to the polynomial which is the ultimate subject of this paper.

Define a subset G of the sphere S^{2m-1} by $G = \{\vec{z} \in S^{2m-1}: \text{for each } j, f_j(\vec{z}) \text{ is a nonnegative real number}\}$. Note that $G \subseteq \overline{F_0} = \{\vec{z} \in S^{2m-1}: f(\vec{z}) \text{ is a nonnegative real number}\}$.

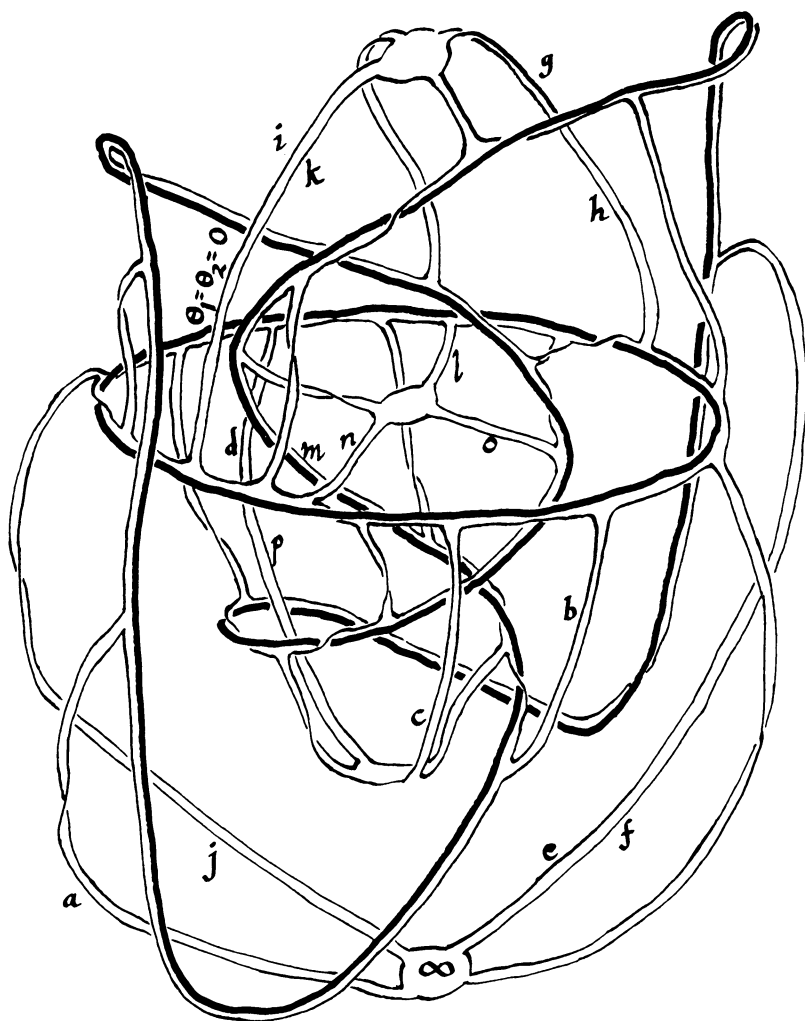


FIGURE 1

It would be useful to be able to assert that G is a deformation retract of \overline{F}_0 . Figures 1 and 2 are psychologically persuasive evidence for this assertion. Figure 1 shows part of the image under the stereographic projection $(z_1, z_2) \mapsto (1 - \text{Im}(z_2))^{-1}(z_1, \text{Re}(z_2))$ from S^3 to $\mathbb{R}^3 \cup \{\infty\}$ of the closure of the fibre F_0 for the polynomial $f(z_1, z_2) = z_1^2 z_2 + z_2^4$. The drawing is not to scale and the point at infinity has been moved into view. The boundary K is in heavy outline. Along the center lines of the bands which are shown connecting the circle $\{z_2 = 0\}$ to either the torus knot $\{z_1^2 + z_2^3 = 0\}$ or one of the four points on the circle $\{z_1 = 0\}$, the arguments of both z_1 and z_2 are constant; if we write $z_j = |z_j| \exp\{i\theta_j\}$, then the bands meet the circle $\{z_2 = 0\}$ at $\theta_1 = k\pi/8$, $k = 0, 1, 2, \dots, 15$. The surface is to be completed by sewing on sixteen disks, whose boundaries are the curves marked "a", "b", "c", \dots , "p". Figure 2 shows the position of the torus knot at a late stage in an evident deformation retraction from \overline{F}_0 to G , which is now also in bold outline.

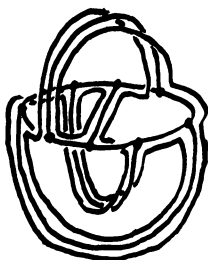


FIGURE 2

I do not know the logical status of the assertion, as I am unable either to prove it or to find a counterexample. Fortunately, for the polynomials which are the subject of this paper, a weaker result will suffice.

DEFORMATION LEMMA. *Under the continuing and additional assumptions above there is a homotopy $H: \overline{F}_0 \times I \rightarrow \overline{F}_0$ such that for all \vec{z} in \overline{F}_0 we have $H(\vec{z}, 0) = \vec{z}$ and $H(\vec{z}, 1) \in G$. Hence if $i: G \rightarrow \overline{F}_0$ denotes the inclusion map, then i_* induces epimorphisms in homology.*

This lemma is true. In fact, I propose to prove it in §§4–14. I hope it is a satisfactory partial response to remark (3.5) on p. 208 of [8].

4. The plan of the proof is to construct H from four homotopies, labeled H_1 to H_4 , and certain other maps designed to correct the target spaces of the H_j 's, viz.,

$$H = g_2 \circ g_1 \circ [H_1 * (\rho \circ (H_2 * H_3))] * H_4.$$

The meaning of the $*$ symbol is as follows. Given spaces X, Y, Z , and W , $Y \subseteq W \subseteq Z$, and homotopies $H': X \times I \rightarrow Y$ and $H'': Z \times I \rightarrow W$ such that $H''(y, 0) = y$ for all y in Y , define $H' * H'': X \times I \rightarrow W$ by

$$\begin{aligned} H' * H''(x, t) &= H'(x, 2t) \quad \text{if } 0 \leq t \leq \frac{1}{2}, \\ &= H''(H'(x, 1), 2t - 1) \quad \text{if } \frac{1}{2} \leq t \leq 1. \end{aligned}$$

In other words, do H' first and then H'' .

Recall that $\overline{F}_0 \subseteq S^{2m-1} \subseteq \mathbb{C}^m$, and that $f = f_1 + f_2 + \cdots + f_r$, each f_j being a monomial.

$H_1: \overline{F}_0 \times I \rightarrow \overline{F}_0$ will have the following properties: for all \vec{z} in \overline{F}_0 , $H_1(\vec{z}, 0) = \vec{z}$ and $f(H_1(\vec{z}, 1)) \geq c$, where c is a positive real number to be determined in §5 below.

$H_2: \mathbb{C}^m \times I \rightarrow \mathbb{C}^m$ will have the following properties: for all \vec{z} in \mathbb{C}^m , $H_2(\vec{z}, 0) = \vec{z}$ and for each $j = 1, 2, \dots, r$ the real part $\operatorname{Re}(f_j(H_2(\vec{z}, t)))$ increases as t increases from 0 to 1 until either $\operatorname{Re}(f_j(H_2(\vec{z}, 1))) > 0$ or $|f_j(H_2(\vec{z}, 1))| < \delta$, where δ is a positive number to be determined in §12 below; the imaginary parts do not change during H_2 .

$H_3: \mathbb{C}^m \times I \rightarrow \mathbb{C}^m$ will have the following properties: for all \vec{z} in \mathbb{C}^m , $H_3(\vec{z}, 0) = \vec{z}$ and for each $j = 1, 2, \dots, r$ the absolute value of the imaginary part $\operatorname{Im}(f_j(H_3(\vec{z}, t)))$ decreases as t increases from 0 to 1 until either $\operatorname{Im}(f_j(H_3(\vec{z}, 1))) = 0$

or $|f_j(H_3(\vec{z}, 1))| < \delta$; the real parts do not change during H_3 . In addition, for all \vec{z} in \mathbb{C}^m such that $\operatorname{Re}(f(\vec{z})) > c$ the point $H_2 * H_3(\vec{z}, t)$ will lie outside S^{2m-1} .

The map $\rho: \mathbb{C}^m - \{0\} \rightarrow S^{2m-1}$ is simply the normalization: $\rho(\vec{z}) = \vec{z}/|\vec{z}|$.

$H_4: S^{2m-1} \times I \rightarrow S^{2m-1}$ will leave the argument of each coordinate z_k (hence of each monomial $f_j(\vec{z})$) unchanged and will reduce the magnitude of each monomial of magnitude δ or less to zero, without allowing any such magnitude to get as large as c/m along the way.

Assuming the above premises fulfilled, we will have found a homotopy $[H_1 * (\rho \circ (H_2 * H_3))] * H_4: \overline{F_0} \times I \rightarrow \{\vec{z} \in S^{2m-1}: \operatorname{Re}(f(\vec{z})) > -c\}$ such that, for all \vec{z} in $\overline{F_0}$, $([H_1 * (\rho \circ (H_2 * H_3))] * H_4)(\vec{z}, 1) \in G$. The final step will be to compose with functions g_1 and g_2 so as to change the target space of the homotopy from $\{\operatorname{Re}(f(\vec{z})) > -c\}$ to $\overline{F_0}$, i.e., to $\{\operatorname{Re}(f(\vec{z})) > 0 \text{ and } \operatorname{Im}(f(\vec{z})) = 0\}$. The map g_1 will satisfy $g_1: \{\vec{z} \in S^{2m-1}: \operatorname{Re}(f(\vec{z})) > -c\} \rightarrow \bigcup_{-\pi < \theta < \pi} \overline{F_\theta} = S^{2m-1} - F_\pi$, and g_2 will "close the book" (see [3, p. 6] or [7, pp. 705–706]).

5. It is convenient to begin at the end, with g_1 and g_2 , since the construction of g_1 dictates a choice of the positive number c . Since $\overline{F_0}$ is a smooth manifold with boundary, the boundary being K , there exists a collaring of K , that is, there is an open neighborhood U of K in $\overline{F_0}$ and a diffeomorphism $\Phi: K \times [0, 1) \rightarrow U$. Hence there is a neighborhood W of K included in U , say $W = \Phi(K \times [0, \frac{1}{2}))$, and a smooth homotopy $P: \overline{F_0} \times I \rightarrow \overline{F_0}$ such that $P(\vec{z}, 0) = \vec{z}$ for all \vec{z} in $\overline{F_0}$, $P(\vec{z}, t) = \vec{z}$ for all \vec{z} in K and all t , and $P(\vec{z}, 1) \in K$ for all \vec{z} in W . Let $c = \min\{|f(\vec{z})| = f(\vec{z}): \vec{z} \in \overline{F_0} - W\}$. Then $|f(\vec{z})| < c$ implies $P(\vec{z}, 1) \in K$. (Alternatively, we may take c to be the minimum critical value of $|f|$ on F_0 and use the Morse theory of $|f|$ to construct P ; compare [3, pp. 49–50].)

Now for any angle θ , let $h_\theta: \overline{F_0} \rightarrow \overline{F_\theta}$ be the diffeomorphism defined by $h_\theta(z_1, z_2, \dots, z_m) = (z_1 \cdot \exp\{i\theta/w_1\}, z_2 \cdot \exp\{i\theta/w_2\}, \dots, z_m \exp\{i\theta/w_m\})$. We will make our two retractions g_1 and g_2 from P by using the family $\{h_\theta\}$.

First, let $X = \bigcup \{\overline{F_\theta}: \pi/2 \leq \theta \leq 3\pi/2\}$ and define $p_1: X \rightarrow X$ by $p_1(\vec{z}) = h_\theta(P(h_{-\theta}(\vec{z}), |\cos \theta|))$ if $\vec{z} \in \overline{F_\theta}$. Then define $g_1: \{\vec{z} \in S^{2m-1}: \operatorname{Re}(f(\vec{z})) > -c\} \rightarrow \bigcup \{\overline{F_\theta}: -\pi < \theta < \pi\}$ by setting $g_1(\vec{z}) = p_1(\vec{z})$ if $\vec{z} \in \bigcup \{\overline{F_\theta}: \pi/2 \leq \theta \leq 3\pi/2\}$ and $g_1(\vec{z}) = \vec{z}$ otherwise.

Second, define $p_2(\vec{z}) = P(\vec{z}, 1)$ for \vec{z} in $\overline{F_0}$ and define $g_2: \bigcup \{\overline{F_\theta}: -\pi < \theta < \pi\} \rightarrow \overline{F_0}$ as follows:

$$\begin{aligned} g_2(\vec{z}) &= h_{(1-u)\theta} \circ p_2 \circ h_{-\theta}(\vec{z}) & \text{if } \vec{z} \in \overline{F_\theta} \text{ and } |f(\vec{z})| = c \cdot u, 0 < u \leq 1; \\ g_2(\vec{z}) &= p_2 \circ h_{-\theta}(\vec{z}) & \text{if } \vec{z} \in \overline{F_\theta} \text{ and } |f(\vec{z})| > c. \end{aligned}$$

I leave it to the reader to verify that the formulas for g_1 and g_2 are unambiguous and yield continuous functions with the desired properties.

6. Construction of H_1 . This is closely related to the collaring step in §5. Making use of the same notation we could for instance define

$$H_1(\vec{z}, t) = \Phi(\vec{z}', \max\{(1-t)u + t/2, u\}) \quad \text{if } \vec{z} = \Phi(\vec{z}', u) \text{ for some } \vec{z}' \text{ in } K \text{ and some } u \text{ in } [0, 1);$$

$$H_1(\vec{z}, t) = \vec{z} \quad \text{otherwise.}$$

7. Construction of H_2 (heuristics). Assume we are given a positive number δ such that $m\delta \leq c$. (See §12 below for the construction of δ .) The homotopy $H_2: \mathbf{C}^m \times I \rightarrow \mathbf{C}^m$ is a flow which integrates a certain system of differential equations. Recall that our polynomial f is a sum of r monomials f_1, f_2, \dots, f_r whose exponent vectors $(e_{j,1}, e_{j,2}, \dots, e_{j,m})$ for $j = 1, 2, \dots, r$ form an independent set. By adding to f a total of $m - r$ more monomials with coefficient $c_j = 0$ we may without altering any of the sets F_θ, K, G , etc., arrange that $r = m$. Let E be the invertible matrix whose entries are the $e_{j,k}$'s, and let $(E^{-1})_{j,k}$ denote a typical entry of the inverse matrix. Fix \vec{z} in \mathbf{C}^m and write $\vec{p}(t)$ for $H_2(\vec{z}, t)$. If we could arrange that the time derivative of $f_j(\vec{p}(t))$ should satisfy $df_j(\vec{p}(t))/dt = |f_j(\vec{p}(t))|^2 \alpha_j$ for some nonnegative real number α_j (depending on \vec{z}), then in unit time the real part of f_j would increase a distance determined by α_j and \vec{z} , while the imaginary part would remain stationary. This leads as we shall see to the system of differential equations

$$\frac{d}{dt} p_j(t) = p_j(t) \sum_{k=1}^m (E^{-1})_{j,k} \alpha_k(\vec{z}) \overline{f_k(\vec{p}(t))},$$

where p_j denotes the j th coordinate of \vec{p} . Unfortunately, for a satisfactory choice of parameters $\{\alpha_k\}$, this system does not have compact support, and so the method which follows is the grubby one of producing explicit solutions.

8. Let us begin by establishing goals for the real parts of the various monomials to reach in unit time, and then computing the associated velocity multipliers $\alpha_k(\vec{z})$.

Choose a positive real number M such that $|f_k(\vec{z})| \leq M$ for all \vec{z} in S^{2m-1} and $k = 1, 2, \dots, m$, and so that $M \geq 2\delta$. Define $g: \mathbf{C} \rightarrow \mathbf{R}$ ("g" for "goal") as follows. For z in \mathbf{C} , write $z = x + iy$, $x, y \in \mathbf{R}$.

If $|y| \leq \delta$ and $x \leq 0$, $g(z) = \max\{x, |y| - \delta\}$.

If $|y| \leq \delta$ and $0 \leq x \leq \delta/2$, $g(z) = x$.

If $|y| \leq \delta$ and $\delta/2 \leq x$, $g(z) = (2M/\delta - 1)x - M + \delta$.

If $|y| \geq \delta$ and $x \leq 0$, $g(z) = |y| - \delta$.

If $|y| \geq \delta$ and $0 \leq x \leq \delta/2$, $g(z) = x + |y| - \delta$.

If $|y| \geq \delta$ and $\delta/2 \leq x$, $g(z) = (2M/\delta - 1)x - M + |y|$.

Here are some elementary properties of g :

(i) g is continuous;

(ii) for all z , $g(z) \geq x$;

(iii) for all z , $g(z) > 0$ or $|(g(z), y)| \leq \delta$;

(iv) $x \geq \delta$ implies $g(z) \geq M$;

(v) if $y = 0$ and $x \neq 0$, then x and $g(z)$ have the same sign.

Define $\alpha: \mathbf{C} \rightarrow \mathbf{R}^+$ as follows:

$\alpha(0) = 0$.

If $z = x + iy \neq 0$, $\alpha(z) = \int_x^{g(z)} du / (u^2 + y^2)$.

This is legal by property (v) above. Continuity of α at 0 follows from the fact that $g(z) = x$ whenever $|x| + |y| \leq \delta/2$. Continuity away from 0 follows from an elementary estimate.

Define $\alpha_k: \mathbf{C}^m \rightarrow \mathbf{R}^+$ by $\alpha_k(\vec{z}) = \alpha(f_k(\vec{z}))$ for $k = 1, 2, \dots, m$.

9. The k th component of H_2 will have the form $r_k \exp\{i\theta_k\}$. Also entering into the formulas will be the arguments $\{\psi_k\}$ of the various monomials; that is, $f_k = |f_k| \exp\{i\psi_k\}$.

Let us begin with $\theta_k: \{\bar{z} \in \mathbb{C}^m: z_k \neq 0\} \times \mathbb{R} \rightarrow \mathbb{R}/2\pi\mathbb{Z}$ for $k = 1, 2, \dots, m$. Write $z_k = |z_k| \exp\{ia_k\}$ and set

$$\theta_k(\bar{z}, t) = a_k - t \sum_{j=1}^m (E^{-1})_{k,j} \cdot \alpha_j(\bar{z}) \cdot \text{Im}(f_j(\bar{z})).$$

If the coefficient c_k is not zero, write $c_k = |c_k| \exp\{ib_k\}$ and define an argument function $\psi_k: \mathbb{C}^m \times \mathbb{R} \rightarrow \mathbb{R}/2\pi\mathbb{Z}$ by

$$\psi_k(\bar{z}, t) = b_k + \sum_{\{l: z_l \neq 0\}} e_{k,l} \cdot \theta_l(\bar{z}, t).$$

Suppose $f_k(\bar{z}) \neq 0$. Then there is no l such that $e_{k,l} > 0$ and $z_l = 0$. Also $c_k \neq 0$, so ψ_k is defined. Furthermore, $f_k(\bar{z}) = |f_k(\bar{z})| \exp\{i\psi_k(\bar{z}, 0)\}$ and $\psi_k(\bar{z}, t) = \psi_k(\bar{z}, 0) - \alpha_k(\bar{z}) \cdot \text{Im}(f_k(\bar{z})) \cdot t$. Observe that if $f_k(\bar{z})$ is a positive real number, then $f_k(\bar{z}) \cdot \alpha_k(\bar{z}) < 1$; in fact, $\alpha_k(\bar{z}) = (1/f_k(\bar{z})) - (1/g(f_k(\bar{z})))$. Also, if $f_k(\bar{z})$ is not real, then the numbers $\sin \psi_k(\bar{z}, t)$ and $\sin \psi_k(\bar{z}, 0)$ have the same sign for all t in $[0, 1]$. To see this, let $f_k(\bar{z}) = x + iy$, $x, y \in \mathbb{R}$. Assume $y \neq 0$. Then

$$\alpha_k(\bar{z}) = y^{-1} \{ \arctan(g(x + iy)/y) - \arctan(x/y) \}$$

so $\alpha_k(\bar{z}) \cdot \text{Im}(f_k(\bar{z})) = (\text{sgn } y) \cdot [\text{the smallest positive angle from } (x, y) \text{ to } (g(x + iy), y)]$. Since $\psi_k(\bar{z}, 0)$ is an argument for $f_k(\bar{z})$, the assertion follows.

Define $r_k: \mathbb{C}^m \times [0, 1] \rightarrow \mathbb{R}^+$ as follows.

If $z_k = 0$, set $r_k(\bar{z}, t) = 0$ for all t . If $z_k \neq 0$ set

$$r_k(\bar{z}, t) = |z_k| \prod_{\{l: f_l(\bar{z}) \in \mathbb{R}\}} [1 - \alpha_l(\bar{z}) \cdot f_l(\bar{z}) \cdot t]^{-(E^{-1})_{k,l}} \\ \prod_{\{l: f_l(\bar{z}) \notin \mathbb{R}\}} [\sin \psi_l(\bar{z}, t) / \sin \psi_l(\bar{z}, 0)]^{-(E^{-1})_{k,l}}.$$

Define $H_2: \mathbb{C}^m \times [0, 1] \rightarrow \mathbb{C}^m$, $H_2 = (H_2^{(1)}, H_2^{(2)}, \dots, H_2^{(m)})$, by setting

$$H_2^{(k)}(\bar{z}, t) = 0 \quad \text{for all } t \text{ if } z_k = 0, \\ H_2^{(k)}(\bar{z}, t) = r_k(\bar{z}, t) \exp\{i\theta_k(\bar{z}, t)\} \quad \text{if } z_k \neq 0.$$

Here are some properties of H_2 .

(i) $H_2(\bar{z}, 0) = \bar{z}$.

(ii) If $f_k(\bar{z}) \in \mathbb{R}$, then $f_k(H_2(\bar{z}, t)) = f_k(\bar{z})/[1 - \alpha_k(\bar{z}) \cdot f_k(\bar{z}) \cdot t]$.

(iii) If $f_k(\bar{z}) \notin \mathbb{R}$, then

$$f_k(H_2(\bar{z}, t)) = f_k(\bar{z}) \cdot [\sin \psi_k(\bar{z}, 0) / \sin \psi_k(\bar{z}, t)] \cdot \exp\{-i\alpha_k(\bar{z}) \cdot \text{Im}(f_k(\bar{z})) \cdot t\} \\ = |f_k(\bar{z})| \exp\{i\psi_k(\bar{z}, t)\} \cdot \sin \psi_k(\bar{z}, 0) / \sin \psi_k(\bar{z}, t) \\ = \text{Im}(f_k(\bar{z})) [\cot \psi_k(\bar{z}, t) + i].$$

Given \bar{z} , denote $H_2(\bar{z}, t)$ by $\bar{p}(t)$, as in §7, and write $\bar{p}(t) = (p_1(t), \dots, p_m(t))$. Then

$$\frac{d}{dt} p_j(t) = p_j(t) \sum_{k=1}^m (E^{-1})_{j,k} \cdot \alpha_k(\bar{z}) \cdot \overline{f_k(\bar{p}(t))}.$$

To see this, just differentiate, observing that by properties (ii) and (iii) above, $\text{Im}(f_k(\vec{p}(t))) = \text{Im}(f_k(\vec{z}))$ for all t . Since the right-hand sides of these differential equations are smooth functions of p_1, \dots, p_m , it follows from the general theory of ordinary differential equations that H_2 is continuous on $\mathbf{C}^m \times [0, 1]$ (see, for example, [2, Chapter IX, Theorem 4, pp. 156–157]).

Following Milnor [3, p. 33], for an analytic function $h(z_1, \dots, z_m)$ define $\overrightarrow{\text{grad}} h = (\overrightarrow{\partial h / \partial z_1}, \dots, \overrightarrow{\partial h / \partial z_m})$, so that the chain rule reads $dh(\vec{p}(t))/dt = \langle d\vec{p}/dt, \overrightarrow{\text{grad}} h \rangle$. An easy calculation now shows that $\langle d\vec{p}/dt, \overrightarrow{\text{grad}} f_k(\vec{p}(t)) \rangle = |f_k(\vec{p}(t))|^2 \cdot \alpha_k(\vec{z})$. Since this is a real number, it follows that it is the rate of change of the real part of $f_k(\vec{p}(t))$ and (a fact already noted above) that $\text{Im}(f_k(\vec{p}(t)))$ does not change with t . The functions α_k were concocted precisely so that in unit time the real part of f_k would reach the goal $g(f_k(\vec{z}))$, defined in §8.

Hence as t goes from 0 to 1, $\text{Re}(f_k(H_2(\vec{z}, t)))$ increases until either it is positive or $|f_k(H_2(\vec{z}, 1))| \leq \delta$.

10. Construction of H_3 . This is very similar to the construction of H_2 . In brief, here are the formulas.

Define $g': \mathbf{C} \rightarrow \mathbf{R}$ as follows:

$$g'(x + iy) = \min\{0, \max\{y, |x| - \delta\}\} \quad \text{if } y \leq 0;$$

$$g'(x + iy) = \max\{0, \min\{y, \delta - |x|\}\} \quad \text{if } y \geq 0.$$

Define $\alpha': \mathbf{C} \rightarrow \mathbf{R}$ by setting $\alpha'(0) = 0$ and $\alpha'(x + iy) = \int_y^{g'(x+iy)} du/(x^2 + u^2)$ if $x + iy \neq 0$. Then set $\alpha'_k(\vec{z}) = \alpha'(f_k(\vec{z}))$. Note that in contrast to the case of H_2 the goals g' may be less than the initial imaginary parts y , and hence the multipliers α'_k may be negative.

Write $z_k = |z_k| \exp\{ia_k\}$ and define $\theta'_k: \{\vec{z} \in \mathbf{C}^m: z_k \neq 0\} \times \mathbf{R} \rightarrow \mathbf{R}/2\pi\mathbf{Z}$ by setting

$$\theta'_k(z, t) = a_k + t \cdot \sum_{j=1}^m (E^{-1})_{k,j} \cdot \alpha'_j(\vec{z}) \cdot \text{Re}(f_j(\vec{z})).$$

If $c_k \neq 0$, write $c_k = |c_k| \exp\{ib_k\}$ and define $\psi'_k: \mathbf{C}^m \times \mathbf{R} \rightarrow \mathbf{R}/2\pi\mathbf{Z}$ by the formula

$$\psi'_k(\vec{z}, t) = b_k + \sum_{\{l: z_l \neq 0\}} e_{k,l} \cdot \theta'_l(\vec{z}, t).$$

Define $r'_k: \mathbf{C}^m \times [0, 1] \rightarrow \mathbf{R}^+$ as follows: if $z_k = 0$, set $r'_k(\vec{z}, t) = 0$ for all t ; if $z_k \neq 0$, set

$$\begin{aligned} r'_k(\vec{z}, t) = & |z_k| \cdot \prod_{\{l: f_l(\vec{z}) \in i\mathbf{R}\}} [1 + i\alpha'_l(\vec{z}) \cdot f_l(\vec{z}) \cdot t]^{-(E^{-1})_{k,l}} \\ & \cdot \prod_{\{l: f_l(\vec{z}) \notin i\mathbf{R}\}} [\cos \psi'_l(\vec{z}, t) / \cos \psi'_l(\vec{z}, 0)]^{-(E^{-1})_{k,l}}. \end{aligned}$$

Define $H_3^{(k)}: \mathbf{C}^m \times [0, 1] \rightarrow \mathbf{C}$ by setting

$$H_3^{(k)}(\vec{z}, t) = 0 \text{ for all } t \quad \text{if } z_k = 0,$$

and

$$H_3^{(k)}(\vec{z}, t) = r'_k(\vec{z}, t) \cdot \exp\{i\theta'_k(\vec{z}, t)\} \quad \text{if } z_k \neq 0.$$

Then set $H_3 = (H_3^{(1)}, H_3^{(2)}, \dots, H_3^{(m)})$.

This time we have the differential equations

$$\frac{\partial}{\partial t} H_3^{(k)}(\vec{z}, t) = H_3^{(k)}(\vec{z}, t) \sum_{l=1}^m i(E^{-1})_{k,l} \cdot \alpha'_l(\vec{z}) \overline{f_l(H_3(\vec{z}, t))},$$

and hence

$$\left\langle \frac{\partial}{\partial t} H_3(\vec{z}, t), \overrightarrow{\text{grad}} f_k(H_3(\vec{z}, t)) \right\rangle = i\alpha'_k(\vec{z}) \cdot |f_k(H_3(\vec{z}, t))|^2.$$

Thus while the real part of $f_k(H_3(\vec{z}, t))$ remains constant in t , the imaginary part has time derivative $\alpha'_k(\vec{z}) \cdot |f_k(H_3(\vec{z}, t))|^2$. Again, α'_k is concocted so that the imaginary part of f_k reaches the goal $g'(f_k(\vec{z}))$; hence for all \vec{z} either $\text{Im}(f_k(H_3(\vec{z}, 1))) = 0$ or $|f_k(H_3(\vec{z}, 1))| \leq \delta$.

11. At this stage we have a homotopy $H_2 * H_3: \mathbf{C}^m \times I \rightarrow \mathbf{C}^m$ which among other things increases the real parts of all the monomials of f , and hence increases the real part of f . Note next that if $f(\vec{z})$ is real and $f(\vec{z}) > c$, then $(H_2 * H_3)(\vec{z}, 1)$ lies outside the sphere S^{2m-1} . To see this, observe that since $m\delta < c$ (see §12), at least one of the monomials $f_k(\vec{z})$ must have real part at least δ . From property (iv) of §8, we have then that $\text{Re}(f_k(H_2(\vec{z}, 1))) \geq M$. Because H_3 does not alter the real parts of the monomials, it is therefore also the case that $\text{Re}(f_k(H_2 * H_3(\vec{z}, 1))) \geq M$, and hence that $|f_k(H_2 * H_3(\vec{z}, 1))| \geq M$. By choice of M , however, $|f_k|$ does not exceed M on S^{2m-1} , and so the assertion follows. Therefore, when we follow $H_2 * H_3$ with the normalization map $\rho: \mathbf{C}^m - \{0\} \rightarrow S^{2m-1}$ defined by $\rho(\vec{z}) = \vec{z}/|\vec{z}|$, and restrict the composite $\rho \circ (H_2 * H_3)$ to pairs (\vec{z}, t) for which $f(\vec{z})$ is real and $f(\vec{z}) > c$, then at the point $\rho \circ (H_2 * H_3)(\vec{z}, 1)$ every monomial f_k either has magnitude less than δ or has positive real part and zero imaginary part.

12. Definition of δ . Denote by S^+ the portion $\{(r_1, r_2, \dots, r_m) \in S^{m-1} \subseteq \mathbf{R}^m: 0 \leq r_k \text{ for all } k\}$ of the (real) $(m-1)$ -sphere. It will be convenient to deal also with the simplex $\Delta = \{(r_1, r_2, \dots, r_m) \in \mathbf{R}^m: 0 \leq r_k \text{ for all } k \text{ and } r_1 + \dots + r_m = 1\}$. Let $\rho_0: \Delta \rightarrow S^+$ be the homeomorphism defined by projecting from the origin. For \vec{r} in S^+ we will be considering real monomial values $|f_k(\vec{r})| = |c_k| \cdot r_1^{e_{k,1}} r_2^{e_{k,2}} \dots r_m^{e_{k,m}}$.

For $k = 1, 2, \dots, m$ define $G_k = \{\vec{r} \in \Delta: |f_k(\rho_0(\vec{r}))| \geq c/m\}$. (The number c was determined in §5 above.) Then G_k is a compact subset of Δ which for each index j such that $e_{k,j} > 0$ does not meet the j th face of Δ , $\{\vec{r} \in \Delta: r_j = 0\}$. Hence there exists a positive number η such that, for each k , G_k is included in the set $\{\vec{r} \in \Delta: r_j \geq \eta \text{ if } e_{k,j} > 0\}$. Call this set $\Delta(k)$; it is a simplex included in and smaller than Δ with faces parallel to those of Δ . We may assume that $\eta < 1/m$, so that for each k , $\Delta(k)$ contains the barycenter of Δ in its interior.

If $c_k \neq 0$, let $\delta_k = \min\{|f_k(\rho_0(\vec{r}))|: \vec{r} \in \Delta(k)\}$. Then $\delta_k > 0$, since $f_k(\rho_0(\vec{r})) \neq 0$ for \vec{r} in $\Delta(k)$. Note also that $\delta_k \leq c/m$. If $c_k = 0$ we do not define δ_k . Let δ be the minimum of the numbers δ_k for those k such that $c_k \neq 0$. Note that $m\delta \leq c$.

Before constructing H_4 , let us record for later use the crucial properties of δ and η . Let \vec{r} denote $(r_1, r_2, \dots, r_m) \in \Delta$.

(i) For $k = 1, 2, \dots, m$, if $|f_k(\rho_0(\vec{r}))| < \delta$, then for some index j we have $e_{k,j} > 0$ and $r_j < \eta$.

(ii) For $k = 1, 2, \dots, m$, if for some index j we have $e_{k,j} > 0$ and $r_j < \eta$, then $|f_k(\rho_0(\vec{r}))| < c/m$.

13. Construction of H_4 . For (z_1, z_2, \dots, z_m) in S^{2m-1} write $r_j = |z_j|$ for each j , so that $(r_1, r_2, \dots, r_m) \in S^+$. The plan is to construct a homotopy $Q = (Q_1, Q_2, \dots, Q_m): \Delta \times I \rightarrow \Delta$ with the properties:

(i) $Q(\vec{r}, 0) = \vec{r}$;

(ii) if $r_j = 0$, then $Q_j(\vec{r}, t) = 0$ for all t ;

(iii) if $r_j < \eta$, then $Q_j(\vec{r}, t)$ decreases as t increases, and so $Q_j(\vec{r}, t) < \eta$ for all t ;

(iv) if $r_j < \eta$, then $Q_j(\vec{r}, 1) = 0$.

Let us connote properties (ii), (iii), and (iv) briefly by saying that Q "wipes out small coordinates".

Once we have Q we can define $H_4: S^{2m-1} \times I \rightarrow S^{2m-1}$ as follows. Write $z_j = r_j \exp\{i\theta_j\}$, using an arbitrary value of θ_j if $r_j = 0$. Then the j th component of $H_4(\vec{z}, t)$ is $\rho_0 \circ Q_j(\rho_0^{-1}(\vec{r}), t) \cdot \exp\{i\theta_j\}$. Property (ii) of Q guarantees the continuity of H_4 . Properties (iii) and (iv), together with the two properties in §12 of the numbers δ and η , imply that H_4 wipes out each monomial which begins with magnitude not exceeding δ , while never allowing the magnitude to reach c/m during the homotopy.

14. The construction of Q is by induction on the skeletons of Δ . Let $\Delta^{(k)}$ denote the k -skeleton. Given a nonempty subset A of the set $\{1, 2, \dots, m\}$ of indices, define the face Δ_A by $\Delta_A = \{\vec{r} \in \Delta: r_j = 0 \text{ if } j \notin A\}$. Let $Q_A: \Delta_A \times I \rightarrow \Delta_A$ be a homotopy beginning at the identity map such that whenever \vec{r} in Δ_A has the property that $r_j < \eta$ for some j in A , then $Q_A(\vec{r}, t)$ lies on the ray from the barycenter of Δ_A through \vec{r} , increasingly far beyond \vec{r} as t increases and on the boundary $\partial\Delta_A$ when $t = 1$. In other words, Q_A is a radial expansion which pushes to the boundary all points that are sufficiently close to the boundary initially. Let us record some elementary properties of Q_A .

(i) Q_A leaves $\partial\Delta_A$ fixed.

(ii) If $\vec{r} \in \Delta_A$ and $r_j < \eta$, then the j th coordinate of $Q_A(\vec{r}, t)$ decreases as t increases. Hence it is less than η for all t .

(iii) If $\vec{r} \in \text{Int } \Delta_A$ and for some j we have $r_j < \eta$, then $Q_A(\vec{r}, 1)$ is in a lower skeleton of Δ than \vec{r} (although if A has more than two members it is not necessarily true that the j th coordinate of $Q_A(\vec{r}, 1)$ is zero).

On the one-skeleton $\Delta^{(1)}$, define $Q^{(1)}$ by setting $Q^{(1)}(\vec{r}, t) = Q_A(\vec{r}, t)$ if $|A| = 2$ and $\vec{r} \in \Delta_A$. Property (i) above assures that this is unambiguous. It is easy to verify that $Q^{(1)}$ wipes out small coordinates in the sense of §13.

Assume that we have defined a homotopy $Q^{(k)}: \Delta^{(k)} \times I \rightarrow \Delta^{(k)}$ which begins at the identity and wipes out small coordinates. Given any subset A of $\{1, 2, \dots, m\}$ with $k+2$ elements, we can extend the restriction $Q^{(k)}|_{\partial\Delta_A}$ to a homotopy

$\Delta_A \times I \rightarrow \Delta_A$ of the identity map by the homotopy extension property of simplicial complexes, and thus we can extend $Q^{(k)}$ to a homotopy $Q': \Delta^{(k+1)} \times I \rightarrow \Delta^{(k+1)}$ of the identity map with the property that Q' maps $\Delta_A \times I$ into Δ_A whenever $|A| = k + 2$. Define $Q'': \Delta^{(k+1)} \times I \rightarrow \Delta^{(k+1)}$ by setting $Q''(\vec{r}, t) = Q_A(\vec{r}, t)$ if $|A| = k + 2$ and $\vec{r} \in \Delta_A$. By property (i) again, Q'' is unambiguously defined by this formula and does not move $\Delta^{(k)}$. Now define $Q^{(k+1)}: \Delta^{(k+1)} \times I \rightarrow \Delta^{(k+1)}$ by $Q^{(k+1)} = Q'' * Q'$ (see §4 for this notation). I leave it to the reader to verify that $Q^{(k+1)}$ wipes out small coordinates.

This completes the inductive step. Setting $Q = Q^{(m)}$ finishes the construction of the homotopy H and thus proves the Deformation Lemma enunciated in §3.

15. For the rest of this paper, assume that

$$f(z_1, z_2, \dots, z_m) = z_1^{a_1} z_2 + z_2^{a_2} z_3 + \dots + z_{m-1}^{a_{m-1}} z_m + z_m^{a_m} z_1$$

and that $a_1 a_2 \dots a_m > 1$. The goal in what follows is to learn enough about the integral homology of the subset G of S^{2m-1} where each monomial of f takes on only nonnegative real values so that we can apply the lemma of §3. The method is to filter the space G by the number of nonzero coordinates and then to study the associated spectral sequence in total degree $m - 1$. It will turn out that the only contributions to $H_{m-1}G$ are in filtration m and, if m is even, $m/2$.

As a first step, let us decompose G as a sort of cell complex, built up by attaching not ordinary cells or simplices but products of simplices and tori. Let us adopt the convention that references to integral subscripts on such variables as z_k, r_k, θ_k , etc., are to be interpreted modulo m . For example, $z_{m+1} = z_1$.

For any positive integer j let Δ^j denote the not-quite-standard j -simplex $\{\vec{r} \in S^j: r_k \geq 0 \text{ for all } k\}$. Also, for any proper subset A of $\{1, 2, \dots, m\}$ define $\Delta_A = \{\vec{r} \in \Delta^{m-1}: r_k = 0 \text{ if } k \notin A\}$. (These definitions are different from the ones adopted in §12 in that the new Δ_A is the image under the radial projection homeomorphism ρ_0 of the old Δ_A .) Then Δ_A is a simplex of dimension $|A| - 1$. Let $\xi_A: \Delta^{|A|-1} \rightarrow \Delta_A$ be the homeomorphism obtained by inserting zeros among the coordinates of $\Delta^{|A|-1}$ at the locations indexed by the complement of A .

For any positive integer j let T^j denote the standard torus $(S^1)^{\times j}$. Given any proper subset A of $\{1, 2, \dots, m\}$, let us agree to say that A has a *block beginning at $k + 1$* if $k \notin A$ but $k + 1 \in A$. For example, if $m \notin A$ but $1 \in A$, then A has a block beginning at 1. Define $T_A = \{\vec{w} \in T^m: \text{for all } k, \text{ if } A \text{ does not have a block beginning at } k + 1 \text{ then } w_{k+1} = w_k^{-a_k}\}$. Let $b(A)$ denote the number of blocks in A , i.e., the number of members $k + 1$ of $\{1, 2, \dots, m\}$, $0 \leq k \leq m - 1$, such that A has a block beginning at $k + 1$. Then T_A is homeomorphic to $T^{b(A)}$. To avoid ambiguities inherent in the cyclic use of subscripts, let us fix a homeomorphism $\eta_A: T^{b(A)} \rightarrow T_A$ defined by sending w in $T^{b(A)}$ to the m -tuple in T_A whose $(k_j + 1)$ st coordinate is w_j , where $0 \leq k_1 < k_2 < \dots < k_{b(A)} \leq m - 1$ and A has a block beginning at $k_j + 1$ for each $j = 1, 2, \dots, b(A)$.

Define $\phi_A: \Delta_A \times T_A \rightarrow S^{2m-1}$ by

$$\phi_A(\vec{r}, \vec{w}) = (r_1 w_1, r_2 w_2, \dots, r_m w_m).$$

The purpose of defining T_A as above is to ensure that each monomial of f will have only nonnegative real values on the image of ϕ_A . Hence actually $\phi_A: \Delta_A \times T_A \rightarrow G$.

Define $\psi_A: \Delta^{|A|-1} \times T^{b(A)} \rightarrow G$ by $\psi_A = \phi_A \circ (\zeta_A \times \eta_A)$. Here are three elementary properties of the maps ψ_A .

- (i) ψ_A is a homeomorphism on $\text{Int } \Delta^{|A|-1} \times T^{b(A)}$.
- (ii) $\psi_A(\text{Int } \Delta^{|A|-1} \times T^{b(A)}) \cap \psi_B(\text{Int } \Delta^{|B|-1} \times T^{b(B)})$ is empty if $A \neq B$.
- (iii) $\{\bar{z} \in G: z_k = 0 \text{ for at least one } k\} = \bigcup \{\text{Image } \psi_A: A \text{ is a proper subset of } \{1, 2, \dots, m\}\}$.

16. Suppose $k \notin A$ and $k + l + 1 \notin A$, but $k + 1 \in A$, $k + 2 \in A, \dots, k + l \in A$, and suppose $l \geq 2$. Let us connote this information by saying that A has a *long block* beginning at $k + 1$ and ending at $k + l$. Assume that $k = k_j$ in the notation of §15, and let $\pi: T^{b(A)} \rightarrow T^{b(A)}$ be the map which sends $(w_1, w_2, \dots, w_{b(A)})$ to $(w_1, \dots, w_{j-1}, w_j^{-a_k+1}, w_{j+1}, \dots, w_{b(A)})$. Assume also that the $(k + 1)$ st coordinate of $\zeta_A(r_1, r_2, \dots, r_{|A|})$ is r_u , that is, ζ_A inserts $k + 1 - u$ zeros ahead of r_u . Define $i_1: \Delta^{|A|-2} \rightarrow \Delta^{|A|-1}$ to be the injection which sends $(r_1, \dots, r_{|A|-1})$ to $(r_1, r_2, \dots, r_{u-1}, 0, r_u, r_{u+1}, \dots, r_{|A|-1})$, and define $i_2: \Delta^{|A|-2} \rightarrow \Delta^{|A|-1}$ to be the injection which sends $(r_1, \dots, r_{|A|-1})$ to $(r_1, r_2, \dots, r_{u+l-2}, 0, r_{u+l-1}, \dots, r_{|A|-1})$. In other words, i_1 inserts a zero in the u th coordinate so that $\zeta_A \circ i_1$ has a zero in the $(k + 1)$ st coordinate, and similarly $\zeta_A \circ i_2$ has a zero in the $(k + l)$ th coordinate. Then as maps from $\Delta^{|A|-2} \times T^b$ to G , where $b = b(A) = b(A - \{k + 1\}) = b(A - \{k + l\})$,

$$\psi_{A-\{k+l\}} \circ (1 \times \pi) = \psi_A \circ (i_1 \times 1) \quad \text{and} \quad \psi_{A-\{k+l\}} = \psi_A \circ (i_2 \times 1).$$

17. Filter G now by defining G_s to be the closed set $\{\bar{z} \in G: z_k \neq 0 \text{ for at most } s \text{ values of } k\}$. Then $G_0 = \emptyset$ and $G_m = G$. Also, if $0 < s < m$, then by §15(iii)

$$G_s - G_{s-1} = \bigcup \{\psi_A(\text{Int } \Delta^{s-1} \times T^{b(A)}): |A| = s\}.$$

By §15(ii), this union is disjoint. Hence if $0 < s < m$, then the integral homology $H_*(G_s, G_{s-1})$ is a direct sum with one summand

$$H_*(\psi_A(\Delta^{s-1} \times T^{b(A)}), \psi_A(\partial \Delta^{s-1} \times T^{b(A)}))$$

for each subset A of $\{1, 2, \dots, m\}$ having s elements. Note that by §15(i), the summand for A is isomorphic via ψ_A to $H_*(\Delta^{s-1} \times T^{b(A)}, \partial \Delta^{s-1} \times T^{b(A)})$. Also, $H_{s-1+b(A)}(\Delta^{s-1} \times T^{b(A)}, \partial \Delta^{s-1} \times T^{b(A)})$ is infinite cyclic. The manifold $\Delta^{s-1} \times T^{b(A)}$ has a standard orientation. Let ι_A denote the image in $H_{s-1+b(A)}(G_s, G_{s-1})$ under ψ_A of the corresponding generator.

Define a proper subset A of $\{1, 2, \dots, m\}$ to be *thick* if A omits no consecutive pair of indices (consecutive in the cyclic sense, of course), that is, if $j \notin A$ implies $j + 1 \in A$. Note that the pair $(\Delta^{s-1} \times T^{b(A)}, \partial \Delta^{s-1} \times T^{b(A)})$ has homology in degree $m - 1$ if and only if A is thick, and has homology in degree $m - 2$ only if A omits at most one consecutive pair of indices.

Suppose A has a long block beginning at $k + 1$ and ending at $k + l$, and $|A| = s$. According to §16, the connecting homomorphism for the exact sequence

of the triple (G_s, G_{s-1}, G_{s-2}) maps ι_A to a sum in which the summand indexed by $A - \{k+1\}$ is $\pm a_{k+1} \iota_{A-\{k+1\}}$ and the summand indexed by $A - \{k+l\}$ is $\pm \iota_{A-\{k+l\}}$.

18. Consider now the spectral sequence of the filtration $\{G_s\}$ in which $E_{s,t}^1 = H_{s+t}(G_s, G_{s-1})$ and $d_{s,t}^1: E_{s,t}^1 \rightarrow E_{s-1,t}^1$ is the connecting homomorphism of the triple (G_s, G_{s-1}, G_{s-2}) . In total degree $s+t = m-1$,

$$E_{s,m-1-s}^1 = \bigoplus \{Z\iota_A: A \text{ is thick and } |A| = s\}.$$

Note that if $s < m/2$, then no thick A of order s exists, so $E_{s,m-1-s}^1 = 0$. If m is even and $s = m/2$, then there exist precisely two thick subsets with s elements, but neither has any long blocks. Hence by a remark in §17, for $r = 1, 2, 3, \dots, d_{m/2, m/2-1}^r$ is the zero homomorphism.

Assume now that $m/2 < s < m$, and that A has s elements and is thick. Then A has at least one long block. Choose such a block, say beginning at $k+1$. Define A' to be $(A \cup \{k\}) - \{k+1\}$. Then A' has s elements, is thick, and has a long block ending at k . Let A'' be derived from A' similarly by altering membership at the beginning of this new long block, and define $A^{(j)} = (A^{(j-1)})'$ inductively in the same way for all positive integers j . Note that $A^{(m-s)} = A + 1 = \{k+1: k \in A\}$. Therefore $A^{((m-s)m)} = A$. Suppose $d_{s,m-1-s}^1(\sum_B \lambda_B \iota_B) = 0$, where B runs over all thick subsets with s elements, and the λ_B 's are integers. In particular the coefficient of $\iota_{A-\{k+1\}}$ in the direct sum expansion of $d_{s,m-1-s}^1 \sum_B \lambda_B \iota_B$ described in §17 must be zero. This says $\pm a_{k+1} \cdot \lambda_A \pm \lambda_{A'} = 0$, i.e., $\lambda_{A'} = \pm a_{k+1} \cdot \lambda_A$. Inductively, then, we see that $\lambda_A = \lambda_{A^{((m-s)m)}} = \pm (a_1 a_2 \cdots a_m)^{m-s} \lambda_A$. Since we are assuming that $a_1 a_2 \cdots a_m > 1$, it follows that $\lambda_A = 0$. There was nothing special about A , however, so actually $d_{s,m-1-s}^1$ is a monomorphism. Therefore if $m/2 < s < m$, then nothing in total degree $m-1$ of filtration s survives to $E_{s,m-1-s}^2$, let alone to E^∞ .

19. Suppose $\vec{z} \in G$ and for all k , $z_k \neq 0$, i.e., $\vec{z} \in G_m - G_{m-1}$. If we write $z_k = r_k \exp\{i\theta_k\}$, then $r_1^2 + r_2^2 + \cdots + r_m^2 = 1$ and $a_k \cdot \theta_k + \theta_{k+1} \equiv 0 \pmod{2\pi Z}$ for all k .

Let $d = a_1 a_2 \cdots a_m + (-1)^{m-1}$, the determinant of the matrix of exponents for the monomials of f . The system of equations modulo $2\pi Z$ for the θ_k 's has precisely d solutions, in each of which each angle θ_k is an integral multiple of $2\pi/d$. To see this, set $\theta_2 = -a_1 \cdot \theta_1$, $\theta_3 = -a_2 \cdot \theta_2$, \dots , $\theta_m = -a_{m-1} \cdot \theta_{m-1}$ and then observe that $-a_m \cdot \theta_m \equiv \theta_1 \pmod{2\pi Z}$ if and only if $(-1)^m a_1 a_2 \cdots a_m \theta_1 = \theta_1 + 2\pi j$ for some integer j .

Denote the simplex in $\overline{G - G_{m-1}}$ on which $\theta_1 \equiv 2\pi j/d$ by σ_j . Then E_{m-1}^1 , which is $H_{m-1}(G, G_{m-1})$, is free on d generators, namely the homology classes $[\sigma_j]$ of the relative cycles σ_j , $j = 1, 2, \dots, d$.

Denote by x_j the difference $[\sigma_j] - [\sigma_{j+1}]$, $j = 1, 2, \dots, m$.

LEMMA. $x_j \in E_{m-1}^2$ and $x_j - x_l \in E_{m-1}^\infty$ for any j and l .

PROOF. It is convenient for the purposes of this proof to use simplicial homology, regarding G as a simplicial complex in which for each proper subset A of

$\{1, 2, \dots, m\}$ the image of ψ_A is a subcomplex, and likewise each σ_j is a subcomplex.

Fix j and write $I = [0, 1]$. Define for each $k = 0, 1, 2, \dots, m-1$ a map $\tau_k: \Delta^{m-2} \times I \rightarrow G_{m-1}$ as a composite $\tau_k = \psi_{\{1, 2, \dots, m\} - \{k\}} \circ (1 \times \alpha)$, in which $1 \times \alpha: \Delta^{m-2} \times I \rightarrow \Delta^{m-2} \times T^1 = \Delta^{m-2} \times S^1$ and $\alpha: I \rightarrow S^1$ is defined by the formula $\alpha(t) = \exp\{(-1)^k a_1 a_2 \cdots a_k (j+t) \cdot 2\pi i/d\}$. In other words, $\tau_k(r_1, r_2, \dots, r_{m-1}, t) = (r_1 w_1, \dots, r_{k-1} w_{k-1}, 0, r_k w_{k+1}, \dots, r_{m-1} w_m)$ where, for $l = 0, 1, 2, \dots, m-1$,

$$w_{k+l+l} = (w_{k+l})^{-a_{k+l}} = \exp\{(-1)^{k+l} a_1 a_2 \cdots a_{k+l} (j+t) \cdot 2\pi i/d\}.$$

Note that if $k+l \geq m$, since $(-1)^m a_1 a_2 \cdots a_m = 1 + (-1)^m d$, it follows that

$$w_{k+l+l} = w_{k+l-m+1} = \exp\{(-1)^{k+l-m} a_1 a_2 \cdots a_{k+l-m} (1 + (-1)^m d) \cdot (j+t) \cdot 2\pi i/d\}.$$

If we impose on $\Delta^{m-2} \times I$ the simplicial structure obtained in the usual way by means of joins, orienting I from 0 to 1, then $\Delta^{m-2} \times I$ becomes a simplicial complex of dimension $m-1$. The sum of all the $(m-1)$ -simplices is a chain in $C_{m-1}(\Delta^{m-2} \times I)$. Denote its image in $C_{m-1}(G_{m-1})$ under $\tau_{k\#}$, after sufficient subdivision and simplicial approximation on $\Delta^{m-2} \times \text{Int } I$, by τ_k as well (a slight abuse of notation). Define a chain δ_j in $C_{m-1}(G_{m-1})$ by $\delta_j = \sum_{k=0}^{m-1} (-1)^k \tau_k$. It is easy to see that $\partial(\sigma_j - \sigma_{j+1} + \delta_j) \in C_{m-2}(G_{m-2})$. Since δ_j has filtration $m-1$, this proves that $[\sigma_j] - [\sigma_{j+1}] \in E_{m-1}^2$.

To prove the second statement, we must be more precise about $\partial(\sigma_j - \sigma_{j+1} + \delta_j)$. If k and k' are integers, write $k \ll k'$ to mean $k+1 < k'$. For each pair k and k' of indices such that $0 \leq k \ll k' \leq m-1$, define maps $H_{k,k'}^{(0)}$ and $H_{k,k'}^{(1)}: \Delta^{m-3} \times I \times I \rightarrow G_{m-2}$ as follows.

$$H_{k,k'}^{(0)}(\vec{r}, t, u) = \psi_{\{1, 2, \dots, m\} - \{k, k'\}}(\vec{r}, \exp\{(-1)^k a_1 a_2 \cdots a_k \cdot u(j+t) \cdot 2\pi i/d\}, \\ \exp\{(-1)^{k'} a_1 a_2 \cdots a_{k'} \cdot u(j+t) \cdot 2\pi i/d\}).$$

$$H_{k,k'}^{(1)}(\vec{r}, t, u) = \psi_{\{1, 2, \dots, m\} - \{k, k'\}} \\ \cdot (\vec{r}, \exp\{(-1)^k a_1 a_2 \cdots a_k \cdot [u(j+t) + t(-1)^m d] \cdot 2\pi i/d\}, \\ \exp\{(-1)^{k'} a_1 a_2 \cdots a_{k'} \cdot u(j+t) \cdot 2\pi i/d\}).$$

(If $k = 0$ and $k' = m-1$, then $b(\{1, 2, \dots, m\} - \{k, k'\}) = 1$ and we suppress the last argument of $\psi_{\{1, 2, \dots, m\} - \{k, k'\}}$ in both maps.)

Impose the join simplicial structure on $\Delta^{m-3} \times I \times I$. Let β_1 in $C_{m-2}(\Delta^{m-3} \times I \times I)$ be the sum of all the $(m-2)$ -simplices in the subcomplex $\Delta^{m-3} \times I \times \{1\}$. Then a straightforward computation shows that

$$\partial(\sigma_j - \sigma_{j+1} + \delta_j) = \sum_{0 \leq k \ll k' \leq m-1} (-1)^{k+k'-1} ((H_{k,k'}^{(0)})_{\#} \beta_1 - (H_{k,k'}^{(1)})_{\#} \beta_1).$$

Now let γ in $C_{m-1}(\Delta^{m-3} \times I \times I)$ be the sum of all the $(m-1)$ -simplices, and let β_0 in $C_{m-2}(\Delta^{m-3} \times I \times I)$ be the sum of all the $(m-2)$ -simplices in the

subcomplex $\Delta^{m-3} \times I \times \{0\}$. A final boundary computation shows that

$$\begin{aligned} \partial \left\{ \sigma_j - \sigma_{j+1} + \delta_j + \sum_{0 \leq k \leq k' \leq m-1} ((H_{k,k'}^{(0)})_{\#} \gamma - (H_{k,k'}^{(1)})_{\#} \gamma) \right\} \\ = \sum_{0 \leq k \leq k' \leq m-1} (-1)^{k+k'-1} ((H_{k,k'}^{(0)})_{\#} \beta_0 - (H_{k,k'}^{(1)})_{\#} \beta_0). \end{aligned}$$

Note that the restrictions of $H_{k,k'}^{(0)}$ and $H_{k,k'}^{(1)}$ to $\Delta^{m-3} \times I \times \{0\}$ are independent of j . Hence modulo chains of filtration lower than m , $(\sigma_j - \sigma_{j+1}) - (\sigma_l - \sigma_{l+1})$ is a permanent cycle. This proves the second statement of the lemma.

20. It will be useful to record part of the result of the last boundary calculation in §19 in the following form.

In $E_{m-2,0}^2$, which, according to §17 is the quotient by the image $d_{m-1,0}^1(\bigoplus_{|A|=m-1} Z_{t_A})$ of a subgroup of

$$\bigoplus_{|B|=m-2} H_{m-2}(\psi_B(\Delta^{m-3} \times T^{b(B)}), \psi_B(\partial\Delta^{m-3} \times T^{b(B)})),$$

$d_{m-1}^2 x_j$ is represented by a class whose component indexed by $B = \{1, 2, \dots, m-2\}$ is $(-1)^m \iota_{\{1, 2, \dots, m-2\}}$, but whose component indexed by $B = \{0, 1, \dots, m-1\} - \{k, k+1\}$ is zero for $k = 0, 1, 2, \dots, m-2$.

LEMMA. *If for some integer λ the multiple λx_j survives to E_{m-1}^3 , then $d|\lambda$.*

PROOF. Assume that $d_{m-1}^2 \lambda x_j = 0$. Then some class in $E_{m-2,0}^1$ whose component indexed by $\{1, 2, \dots, m-2\}$ is $(-1)^m \lambda \iota_{\{1, 2, \dots, m-2\}}$ and whose component indexed by $\{0, 1, 2, \dots, m-1\} - \{k, k+1\}$ is zero for $k = 0, 1, 2, \dots, m-2$ actually belongs to the image of $d_{m-1,0}^1$; say it is $d^1 \sum_{k=1}^m \lambda_k \iota_{\{1, 2, \dots, m\} - \{k\}}$. The rest of the argument is similar to the end of §18. According to the result at the end of §17, refined to take careful account of signs, $\lambda_{k+1} = a_{k+1} \cdot \lambda_k$ for $k = 1, 2, \dots, m-2$,

$$\lambda_1 = (-1)^m a_1 \cdot \lambda_m \quad \text{and} \quad (-1)^m (a_m \cdot \lambda_{m-1} - \lambda_m) = (-1)^m \lambda.$$

Hence $\lambda = (-1)^m \lambda_0 \cdot d$. This proves the lemma.

21. We can now compute $E_{s,t}^{\infty}$ when $s+t = m-1$. First recall that E_{m-1}^1 , which is $H_{m-1}(G, G_{m-1})$, is free abelian of rank d on the basis $[\sigma_j]$, $j = 1, 2, \dots, d$. The map $(z_1, z_2, \dots, z_m) \mapsto (|z_1|, |z_2|, \dots, |z_m|)$ sends the pair (G, G_{m-1}) in \mathbf{C}^m to the simplex pair $(\Delta^{m-1}, \partial\Delta^{m-1})$ in \mathbf{R}^m . In fact, this map induces a morphism of the spectral sequence $\{E'_{*,*}\}$ into the spectral sequence associated with the corresponding filtration (by the number of nonzero variables) on Δ^{m-1} . Applying this morphism to $d^1: E_{m-1}^1 \rightarrow E_{m-1,-1}^1$, we see that no integral combination $\sum_{j=1}^d \lambda_j [\sigma_j]$ can survive to E_{m-1}^2 unless $\sum_{j=1}^d \lambda_j = 0$. According to the lemma of §19, however, $x_j = [\sigma_j] - [\sigma_{j+1}]$ does survive, and hence the survivors to E_{m-1}^2 are precisely the integral span of the classes x_j , $j = 1, 2, \dots, d-1$.

The weights w_1, w_2, \dots, w_m associated with the weighted homogeneous polynomial f satisfy $a_j/w_j + 1/w_{j+1} = 1$ for all j . Let

$$\begin{aligned} n_j = (-1)^{m-1} + (-1)^{m-2} a_{j-1} + (-1)^{m-3} a_{j-2} a_{j-1} + \dots \\ + (-1)^{m-k} a_{j-k+1} a_{j-k+2} \dots a_{j-1} + \dots + a_1 a_2 \dots a_{j-1} a_{j+1} \dots a_m. \end{aligned}$$

Then it is easy to see that $1/w_j = n_j/d$. According to Milnor and Orlik [4], the rank μ of $H_{m-1}(\overline{F}_0)$ is $(w_1 - 1)(w_2 - 1) \cdots (w_m - 1)$. An easy computation shows that $\mu = a_1 a_2 \cdots a_m$. Hence $\mu = d - 1$ if m is odd and $\mu = d + 1$ if m is even.

According to §18, $E_{s,m-s-1}^2 = 0$ unless $s = m$ or m is even and $s = m/2$. In the latter case, $E_{m/2,m/2-1}^2$ has rank at most 2. Since, by the Deformation Lemma of §3, $i_*: H_{m-1}G \rightarrow H_{m-1}\overline{F}_0$ is an epimorphism, it follows that it is actually an isomorphism, that $E_{m,-1}^\infty$ has rank $d - 1$, and that when m is even $E_{m/2,m/2-1}^\infty$ has rank 2.

LEMMA. $E_{m,-1}^\infty$ is free abelian, generated by the $d - 1$ differences $x_1 - x_2, x_2 - x_3, \dots, x_{d-1} - x_d$.

PROOF. Because of the rank considerations above, we need only establish that the given differences generate $E_{m,-1}^\infty$. We know that $E_{m,-1}^2$ is spanned by the x_j 's. Suppose $\sum_{j=1}^d \lambda_j x_j \in E_{m,-1}^\infty$. Then, by the lemma of §19, $(\sum_{j=1}^d \lambda_j) x_1 \in E_{m,-1}^\infty$. It follows then from the lemma of §20 that $d | \sum_{j=1}^d \lambda_j$. But $0 = x_1 + x_2 + \cdots + x_d$, and so, after subtracting the appropriate multiple of this expression, we may assume that $\sum_{j=1}^d \lambda_j = 0$. This proves the lemma.

22. It remains to compute the action of the characteristic map h_* in $E_{*,*}^\infty$, using the formula

$$h(z_1, z_2, \dots, z_m) = (z_1 \cdot \exp\{2\pi i/w_1\}, z_2 \cdot \exp\{2\pi i/w_2\}, \dots, z_m \cdot \exp\{2\pi i/w_m\}).$$

In case m is even, the two cells in filtration $m/2$ which carry homology of total degree $m - 1$ are actually spheres on which h is homotopic to the identity map. Hence we will concentrate on $E_{m,-1}^\infty$.

From $1/w_j = n_j/d$ we have $a_j n_j + n_{j+1} = d$ for all j , and it follows that the greatest common divisor $\gcd(n_j, d)$ is independent of j , say $\gcd(n_j, d) = q$ for all j . Since $h_\# \sigma_j = \sigma_{j+n_1}$, it follows that $h_*(x_j - x_{j+1}) = x_{j+n_1} - x_{j+n_1+1}$. Thus there is a basis for $E_{m,-1}^\infty$ in which the matrix for h_* is obtained from the $d \times d$ block-diagonal matrix having q nonzero blocks, each of them the $d/q \times d/q$ cyclic permutation matrix

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

by deleting the last column and last two rows, then putting a row of (-1) 's at the bottom.

It is now easy to see that $\ker(I_* - h_*)$, which is isomorphic to $H_{m-1}(K)$, is free abelian of rank $q + (-1)^m$, while $\text{coker}(I_* - h_*)$, which is isomorphic to $H_{m-2}(K)$, is the direct sum of a free abelian group of rank $q + (-1)^m$ and a cyclic group of order d/q .

Let H be the matrix for h_* described above. Then it is also easy to see that $tI - H$ can be diagonalized by row and column operations as a matrix over the polynomial ring $Z[t]$ (even though $Z[t]$ is not a principal ideal domain); that is,

there exist matrices $U(t)$ and $V(t)$ unimodular in the matrix ring of $Z[t]$ for which $U(t)[tI - H]V(t)$ is a diagonal matrix. This verifies for the polynomial f a conjecture of Orlik [6, p. 265].

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DEPARTMENT OF MATHEMATICS, AUGSBURG COLLEGE, MINNEAPOLIS, MINNESOTA 55454